PERGAMON

International Journal of Solids and Structures 39 (2002) 4615–4625

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Stress functions for a heterogeneous section of a tree

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Received 4 March 2002; received in revised form 16 May 2002

**Abstract**

Consider a tree that is composed of wood, which is orthotropic with respect to the cylindrical coordinate axes, where the $z$-axis is directed up the center of the tree. When a section of a tree is considered as a Relaxed Saint-Venant's Problem the stresses in the plane of a transverse cross section will equal zero. By allowing some of, or all of the compliance coefficients to be functions of the radial coordinate $r$, the structure of the stress functions can be fundamentally altered. If the compliance coefficient in the $z$-direction ($S_{3333}$) is allowed to remain constant, the $S_{rz}$ and $S_{\theta z}$ shear stresses will be functions of the coordinates, dimensions, applied loads, and compliance coefficients, while the other stresses will only be functions of the coordinates, dimensions and applied loads. If $S_{3333}$ is also allowed to be a function of $r$, then all the nonzero stresses will be functions of the coordinates, dimensions, applied loads, and compliance coefficients.


*Keywords:* Elasticity; Orthotropic; Heterogeneous; Wood; Beam

## 1. Introduction

Lyons et al. (2002a) derived the stress and displacement equations for a cylindrical section of a tree subject to loads independent of $x_3$, and with compliance coefficients (in cylindrical coordinates) that were constant throughout the section. The constraints placed on the compliance coefficients when assuming they were constant, produced a compliance tensor that acted similarly to one for a transversely isotropic material. Lyons et al. (2002b) noted that the published values for the compliance tensor for *Pseudotsuga menziesii* (Douglas fir) precludes the possibility of the compliance tensor being constant in the cross section of a tree. Lyons et al. (2002b) proposed a compliance tensor that was linearly dependent on the radial coordinate $r$.

The objectives of this paper are to derive the stress equations for a cylindrical section of a tree for two different constitutive equations that allow for different levels of heterogeneity. The first constitutive equation considered will be similar to the one proposed by Lyons et al. (2002b) where only the minimum

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PII: S0020-7683(02)00381-5

dependence on $r$ was assumed in order to allow for the nonunique strains at $r = 0$. The second constitutive equation to be considered is similar to the constitutive equation proposed by Lyons et al. (2002b), except that the compliance and elasticity coefficients in the $z$-direction will also be linear functions of the radial coordinate $r$.

## 2. Constitutive equations and problem statement

The following constitutive equation in cylindrical coordinates was proposed by Lyons et al. (2002b) and will be used to model a cylindrical section of a tree (prime denotes in cylindrical coordinates).

$$\left.\begin{aligned} E'_{ij} &= S'_{ijkl}S'_{kl} = \left\lfloor \underline{S_{ijkl}} + r^* \underline{M_{ijkl}} \right\rfloor S'_{kl} \\ S'_{ij} &= C'_{ijkl}E'_{kl} = \left[ \underline{C_{ijkl}} + r^* \underline{K_{ijkl}} \right] E'_{kl} \end{aligned}\right\} \text{in cylindrical coordinates} \tag{2.1}$$

Here $E'_{ij}$ and $S'_{ij}$ are the infinitesimal strain tensor and Cauchy's stress tensor, $S'_{ijkl}$ and $C'_{ijkl}$ are the compliance and elasticity tensors, and $\underline{S_{ijkl}}, \underline{M_{ijkl}}, \underline{C_{ijkl}}$, and $\underline{K_{ijkl}}$ are constants.

Lyons et al. (2002a) transformed the elasticity and compliance tensors from a cylindrical basis to a Cartesian basis,

$$\begin{aligned} C_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}C'_{mnrs} \\ S_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}S'_{mnrs} \end{aligned} \tag{2.2}$$

where $Q_{ij} = Q_{ij}(\theta)$ is a clockwise rotation about the $x_3$-axis. The complete list of transformation equations is included in Appendix A. Given (2.2) the constitutive equations can be written in Cartesian coordinates

$$S_{ij} = C_{ijkl}E_{kl}, \quad E_{ij} = S_{ijkl}S_{kl} \tag{2.3}$$

Lyons et al. (2002b) showed the transformed compliance tensor could be simplified so that some of the coefficients could remain constant, while still satisfying the constraints imposed by the nonunique strains in the $\mathbf{e}_\theta$ and $\mathbf{e}_r$ directions. The simplified equations for the transformed compliance tensor are included in Appendix B.

With the compliance coefficients transformed into Cartesian coordinates, Eq. (B.1), it is possible to use Iesan's (1987) formulation to solve the relaxed Saint-Venant's problem. Lyons et al. (2002a) considered a cylindrical section of a tree as a relaxed Saint-Venant's problem with loads independent of $x_3$. The statement of the relaxed Saint-Venant's problem will be repeated here for convenience.

From now on in this paper, Greek indices will range from 1 to 2, while Latin indices range from 1 to 3 unless otherwise specified. Summation notation is used for repeated indices, and a comma followed by a subscript will indicate a partial derivative with respect to the coordinate. Note the following special notations will be used, the Kronecker delta function ($\delta_{ij}$), and the two-dimensional alternator symbol ($e_{\alpha\beta}$).

Consider a cylindrical section of a tree as a cantilever beam with constant cross sections (Fig. 1). Let $\Sigma_1$ be the open cross section at $x_3 = 0$, let $\Sigma_2$ be the open cross section at $x_3 = h$, and let $\Sigma$ be an arbitrary open cross section with normal $x_3$. The lateral surface of the cylinder will be $\Pi$, while the boundary of an arbitrary cross section is $\Gamma$. The closure of an arbitrary cross section will be $\overline{\Sigma} = \Sigma \cup \Gamma$.

The resultant loads applied to the cross section at $x_3 = 0$ are the forces $\mathbf{F}$ and the moments $\mathbf{M}$, and these are represented by integral functions of the displacement vector $\mathbf{u}$, where $f(\mathbf{u}) = \mathbf{F}$ and $m(\mathbf{u}) = \mathbf{M}$. The lateral surface of the cylinder is unloaded, the cross section at $\Sigma_2$ is fixed, and body loads will be ignored in this analysis. The problem in Fig. 1 is of the class $P_1$ as defined by Iesan (1987), where the resultant loads acting on $\Sigma$ are independent of $x_3$ and $F_\alpha = 0$.

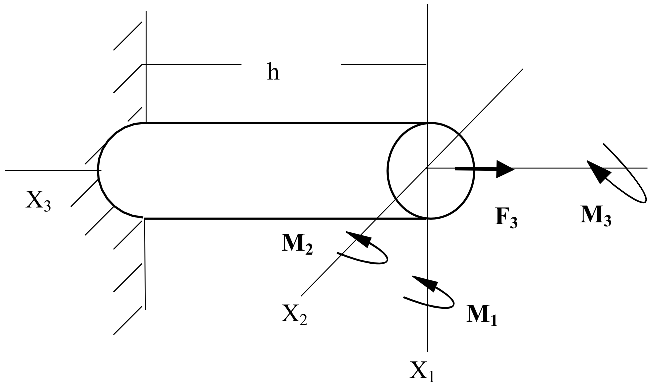


Fig. 1. Cylindrical cantilever beam subject to loads independent of $x_3$.

Lyons et al. (2002a) presented the total displacements as

$$u_i^0 = u_i + u_i^I \tag{2.4}$$

where $u_i$ are the displacements resulting from strain, and $u_i^I$ are displacement resulting from a rigid body motion. The displacements resulting from strain, derived in a manner similar to that used by Iesan (1987), are

$$u_i = \delta_{i\alpha}\left[ -a_\alpha \frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_\beta x_3 \right] + \delta_{i3}[a_\rho x_\rho + a_3]x_3 + W_i \tag{2.5}$$

where $a_p$ are constants that will have to be determined using the boundary conditions, and $\mathbf{W} = \mathbf{W}(x_1, x_2)$ is a vector composed of the functions of integration.

Since the body forces are being ignored and the lateral surface of the cylinder is unloaded, the necessary conditions for a solution imply that the sum of the stress fields acting on $\Sigma_2$ must be in equilibrium with the resultant loads acting on $\Sigma_1$ therefore,

$$\begin{aligned} &\int_{\Sigma_2} S_{\alpha 3}(\mathbf{u})\,\mathrm{d}a = -f_\alpha(\mathbf{u}) = 0, \quad \int_{\Sigma_2} S_{33}(\mathbf{u})\,\mathrm{d}a = -f_3(\mathbf{u}) = -F_3, \\ &\int_{\Sigma_2} e_{\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u})\,\mathrm{d}a = -m_3(\mathbf{u}) = -M_3, \quad \int_{\Sigma_2} x_\alpha S_{33}(\mathbf{u})\,\mathrm{d}a = e_{\alpha\beta} m_\beta(\mathbf{u}) = e_{\alpha\beta} M_\beta \end{aligned} \tag{2.6}$$

Substitute (2.5) into the definition of the infinitesimal strain tensor, the resulting strains are

$$\begin{aligned} &E_{11}(\mathbf{u}) = W_{1,1}, \quad E_{22}(\mathbf{u}) = W_{2,2}, \quad E_{33}(\mathbf{u}) = (a_\rho x_\rho + a_3), \quad E_{23}(\mathbf{u}) = \tfrac{1}{2}[a_4 x_1 + W_{3,2}], \\ &E_{13}(\mathbf{u}) = \tfrac{1}{2}[-a_4 x_2 + W_{3,1}], \quad E_{12}(\mathbf{u}) = \tfrac{1}{2}[W_{1,2} + W_{2,1}] \end{aligned} \tag{2.7}$$

Substitute the strain tensor (2.7) into the constitutive equation (2.3), then the stress tensor in Cartesian coordinates becomes

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W}) \tag{2.8}$$

The $T_{ij}(\mathbf{W}) = C_{ijk\alpha} W_{k,\alpha}$, are the stresses resulting from the displacement vector $\mathbf{W}$, which is independent of $x_3$ and so forms a generalized plain strain problem. Iesan (1987) found that the generalized plane strain problem could be separated into four auxiliary problems $T_{ij}^{(p)}$ $(p = 1, 2, 3, 4)$, which are defined by the following equilibrium equations (1)–(3) and boundary conditions (4)–(6).

$$\begin{aligned}
&(1)\ T^{(\beta)}_{i\alpha}(\mathbf{W})_{,\alpha} + (C_{i\alpha33}x_\beta)_{,\alpha} = 0\\
&(2)\ T^{(3)}_{i\alpha}(\mathbf{W})_{,\alpha} + (C_{i\alpha33})_{,\alpha} = 0\\
&(3)\ T^{(4)}_{i\alpha}(\mathbf{W})_{,\alpha} - e_{\rho\beta}(C_{i\alpha\rho3}x_\beta)_{,\alpha} = 0\\
&(4)\ T^{(\beta)}_{i\alpha}(\mathbf{W})n_\alpha = -C_{i\alpha33}x_\beta n_\alpha\\
&(5)\ T^{(3)}_{i\alpha}(\mathbf{W})n_\alpha = -C_{i\alpha33}n_\alpha\\
&(6)\ T^{(4)}_{i\alpha}(\mathbf{W})n_\alpha = e_{\rho\beta}C_{i\alpha\rho3}x_\beta n_\alpha
\end{aligned} \tag{2.9}$$

Here **n** is the unit normal to $\Gamma$. The auxiliary problems combine as follows:

$$T_{ij}(\mathbf{W}) = \sum_{p=1}^{4} a_p T^{(p)}_{ij}(\mathbf{W}) \tag{2.10}$$

After substituting the stresses (2.8) into the necessary conditions for a solution (2.6), and taking note of the simplifications resulting from (B.1), the following system of equations can be found for determining $a_p$.

$$\begin{bmatrix} \int_{\Sigma_2} x_1^2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_1x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_1 C_{3333}\,\mathrm{d}a & 0\\ \int_{\Sigma_2} x_1x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2^2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2 C_{3333}\,\mathrm{d}a & 0\\ \int_{\Sigma_2} x_1 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} C_{3333}\,\mathrm{d}a & 0\\ 0 & 0 & 0 & \int_{\Sigma_2} [G_5]\,\mathrm{d}a \end{bmatrix} \begin{bmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{bmatrix} = \begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4 \end{bmatrix} \tag{2.11}$$

where

$$\begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4 \end{bmatrix} = \begin{bmatrix} M_1 - \int_{\Sigma_2} x_1 T_{33}\,\mathrm{d}a\\ M_2 - \int_{\Sigma_2} x_2 T_{33}\,\mathrm{d}a\\ -F_3 - \int_{\Sigma_2} T_{33}\,\mathrm{d}a\\ -M_3 - \int_{\Sigma_2} x_1 T_{32} + x_2 T_{31}\,\mathrm{d}a \end{bmatrix} \quad \text{and} \quad G_5 = [-2x_1x_2C_{1323} + x_1^2C_{2323} + x_2^2C_{1313}]$$

Note from (B.1) that $C_{3333} = C'_{3333}$, where $C_{3333}$ is a constant, and recall that the integrals are taken over a circular cross-section. Therefore, equation (2.11) becomes

$$\begin{bmatrix} C'_{3333}I & 0 & 0 & 0\\ 0 & C'_{3333}I & 0 & 0\\ 0 & 0 & C'_{3333}A & 0\\ 0 & 0 & 0 & \int_{\Sigma_2}[G_5]\,\mathrm{d}a \end{bmatrix} \begin{bmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{bmatrix} = \begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4 \end{bmatrix} \tag{2.12}$$

Here $I$ is the moment of inertia, and $A$ is the cross-sectional area. Since $C'_{3333}$, $A$ and $I$ are never zero, then $a_p$ ($p = 1, 2, 3$) are uniquely defined by (2.12). The integrand $G_5$ can be simplified as follows when taking (B.1) into account,

$$\begin{aligned} G_5 &= \left[-2x_1x_2C_{1323} + x_1^2C_{2323} + x_2^2C_{1313}\right]\\ &= r^3[2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta) + \sin^2(\theta)]\underline{M_{2323}} + r^2[\cos^2(\theta) + \sin^2(\theta)]\underline{C_{1313}}\\ &= r^3\underline{M_{2323}} + r^2\underline{C_{1313}} \end{aligned} \tag{2.13}$$

Note in Eq. (2.13) that it is possible for $\underline{M_{2323}}$ to be negative, where as $\underline{C_{1313}}$ is always positive. However, trees in second growth stands are commonly less than one meter in diameter, then $r^3 < r^2$. Therefore, when $|\underline{M_{2323}}| \leqslant |\underline{C_{1313}}|$, (2.13) will be greater than zero except at $r = 0$, and so $a_4$ is uniquely defined by (2.12) except possibly at $r = 0$.

## 3. Stress functions when considering $S_{3333}$ a constant

Expand Eq. (2.8),

$$\begin{aligned}
&(1)\ S_{11} = C_{1133}(a_1x_1 + a_2x_2 + a_3) + T_{11}(\mathbf{W}) \\
&(2)\ S_{22} = C_{2233}(a_1x_1 + a_2x_2 + a_{32}) + T_{22}(\mathbf{W}) \\
&(3)\ S_{33} = C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}(\mathbf{W}) \\
&(4)\ S_{23} = C_{2332}(a_4x_1) - C_{2313}(a_4x_2) + T_{23}(\mathbf{W}) \\
&(5)\ S_{13} = C_{1332}(a_4x_1) - C_{1313}(a_4x_2) + T_{13}(\mathbf{W}) \\
&(6)\ S_{12} = C_{1233}(a_1x_1 + a_2x_2 + a_3) + T_{12}(\mathbf{W})
\end{aligned} \tag{3.1}$$

Lyons et al. (2002b) showed that the in-plane stresses in cylindrical coordinates ($S'_{\alpha\beta}$) are functions of the in-plane stresses in Cartesian coordinates ($S_{\alpha\beta}$).

$$S'_{\alpha\beta} = Q_{\alpha\rho}Q_{\beta\gamma}S_{\rho\gamma} \tag{3.2}$$

Note in (3.2) that $S'_{\alpha\beta}$ and $S_{\alpha\beta}$ are $2 \times 2$ second order tensors, and that $Q_{\alpha\beta}$ is the transformation

$$Q_{\alpha\beta} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3.3}$$

Therefore, since (3.3) is an orthonormal transformation the in-plane stresses in Cartesian coordinates ($S_{\alpha\beta}$) can be written as functions of the in-plane stresses in cylindrical coordinates ($S'_{\alpha\beta}$),

$$\begin{aligned}
Q^{-1}_{\beta\gamma}Q^{-1}_{\alpha\rho}S'_{\alpha\beta} &= Q^{-1}_{\beta\gamma}Q^{-1}_{\alpha\rho}Q_{\alpha\rho}Q_{\beta\gamma}S_{\rho\gamma} \\
S_{\rho\gamma} &= Q^{-1}_{\beta\gamma}Q^{-1}_{\alpha\rho}S'_{\alpha\beta}
\end{aligned} \tag{3.4}$$

where $Q^{-1}_{\alpha\beta}$ is the inverse of $Q_{\alpha\beta}$.

Lyons et al. (2002b) proved for problems where the applied loads are independent of $x_3$ that $S'_{\alpha\beta} = 0$ and so by (3.4)

$$S_{\rho\gamma} = 0 \tag{3.5}$$

Substitute (3.5) into (3.1), then the first, second and sixth equations become

$$\begin{aligned}
T_{11} &= -C_{1133}(a_\rho x_\rho + a_3) \\
T_{22} &= -C_{2233}(a_\rho x_\rho + a_3) \\
T_{12} &= -C_{1233}(a_\rho x_\rho + a_3)
\end{aligned} \tag{3.6}$$

Recall the displacement vector $\mathbf{W} = \mathbf{W}(x_1, x_2)$, then

$$E_{33}(\mathbf{W}) = S_{3311}T_{11} + S_{3322}T_{22} + S_{3333}T_{33} + 2S_{3312}T_{12} = 0 \tag{3.7}$$

Solve (3.7) for $T_{33}$, then

$$T_{33} = -\frac{1}{S_{3333}}[S_{3311}T_{11} + S_{3322}T_{22} + 2S_{3312}T_{12}] \tag{3.8}$$

Recall for generalized plane strain that $E_{ij} = S_{ijmn}T_{mn}$. Chirita (1979) notes this process must be reversible, therefore, $T_{kl} = C_{klrs}E_{rs}$. This results in $E_{ij} = S_{ijmn}C_{mnrs}E_{rs}$, and

$$S_{ijmn}C_{mnrs} = \frac{1}{2}[\delta_{ir}\delta_{js} + \delta_{ij}\delta_{jr}] = \begin{cases} (i = s, j = r) \Rightarrow 1/2 \\ (i = r, j = s) \Rightarrow 1/2 \\ (i = r, j = s; i = s, j = r) \Rightarrow 1 \\ \text{all other } i, j, r, s \Rightarrow 0 \end{cases} \tag{3.9}$$

Substitute (3.6) into (3.8) while taking (3.9) into account, then

$$T_{33} = [a_\rho x_\rho + a_3]\left[S^{-1}_{3333} - C_{3333}\right] \tag{3.10}$$

Following procedures similar to those used by Lyons et al. (2002a) to determine the auxiliary generalized plane strain stress functions, it can be shown that

$$T^{(p)}_{\alpha 3} = 0 \quad \text{for} \quad p = 1, 2, 3 \tag{3.11}$$

Therefore, by (2.10)

$$T_{\alpha 3}(\mathbf{W}) = a_4 T^{(4)}_{\alpha 3}(\mathbf{W}) \tag{3.12}$$

Recall the third and sixth equations of (2.9)

$$\begin{aligned} T^{(4)}_{13},_1 + T^{(4)}_{23},_2 &= -x_1(C_{1323},_1 + C_{2323},_2) + x_2(C_{1313},_1 + C_{1323},_2) \\ T^{(4)}_{13} n_1 + T^{(4)}_{23} n_2 &= (C_{1313}x_2 - C_{1323}x_1)n_1 + (C_{1323}x_2 - C_{2323}x_1)n_2 \end{aligned} \tag{3.13}$$

Substitute $C_{\alpha 3 \beta 3}$ from (B.1) into (3.13), then

$$\begin{aligned} T^{(4)}_{13},_1 + T^{(4)}_{23},_2 &= 4r\left[\underline{M_{1313}} - \underline{M_{2323}}\right] \cos(\theta)^3 \sin(\theta) \\ T^{(4)}_{13} n_1 + T^{(4)}_{23} n_2 &= \frac{r}{2}\left[\underline{M_{1313}} - \underline{M_{2323}}\right] \sin(4\theta) \end{aligned} \tag{3.14}$$

Let $\theta = \frac{m\pi}{2}$, where $m = 1, 2, \ldots$, then (3.14) becomes

$$\begin{aligned} T^{(4)}_{13},_1 + T^{(4)}_{23},_2 &= 0 \\ T^{(4)}_{13} n_1 + T^{(4)}_{23} n_2 &= 0 \end{aligned} \tag{3.15}$$

The auxiliary generalized plane strain stresses in cylindrical coordinates are

$$\begin{aligned} T^{(4)\prime}_{13} &= \cos(\theta) T^{(4)}_{13} + \sin(\theta) T^{(4)}_{23} \\ T^{(4)\prime}_{23} &= -\sin(\theta) T^{(4)}_{13} + \cos(\theta) T^{(4)}_{23} \end{aligned} \tag{3.16}$$

Solve the second equation of (3.15) for $T^{(4)}_{13}$ and substitute this into the first of (3.16), and note that $n_1 = \cos(\theta)$ and $n_2 = \sin(\theta)$, then on $\Gamma$ for $\theta = (m\pi/2)$

$$T^{(4)\prime}_{13} = \cos(\theta)\left(\frac{-T^{(4)}_{23}\sin(\theta)}{\cos(\theta)}\right) + \sin(\theta) T^{(4)}_{23} = 0 \tag{3.17}$$

Substitute (3.17) back into (3.15), then on $\Gamma$ for $\theta = (m\pi/2)$

$$T^{(4)\prime}_{23} = 0 \tag{3.18}$$

Substitute (3.17) and (3.18) into (3.16), then on $\Gamma$ for $\theta = (m\pi/2)$

$$T^{(4)}_{13} = T^{(4)}_{23} = 0 \tag{3.19}$$

Substitute (3.19) into (3.12), then on $\Gamma$ for $\theta = (m\pi/2)$

$$T_{13} = T_{23} = 0 \tag{3.20}$$

Lyons et al. (2002b) proved that the generalized plane stain stresses $T_{ij}$ are independent of the cylindrical coordinate $\theta$, therefore, by (3.20) $T_{13} = T_{23} = 0$ on $\Gamma$. Lyons et al. (2002b) proved that the generalized plane strains could be represented by functions of the complex variable $w = x_1 + ix_2$. Therefore, by Cauchy's integral formula, if $T_{\alpha 3} = 0$ on $\Gamma$, then

$$T_{\alpha 3} = 0 \text{ on } \overline{\Sigma} = \Sigma \cup \Gamma \tag{3.21}$$

The $S_{ij}$ stress functions may be formed by taking note of (3.5) and by substituting (3.10) and (3.21) into the third, fourth, and fifth equations of (3.1), then

$$\begin{aligned} S_{11} &= S_{22} = S_{12} = 0 \\ S_{33} &= S_{3333}^{-1}[a_\rho x_\rho + a_3] \\ S_{23} &= a_4 C_{2323} x_1 - a_4 C_{1323} x_2 \\ S_{13} &= a_4 C_{1323} x_1 - a_4 C_{1313} x_2 \end{aligned} \tag{3.22}$$

Substitute (3.22) into the necessary conditions for a solution, then the first two equations of (2.6) can be shown to equal zero for all $a_4$.

For example

$$\begin{aligned} \int_0^{2\pi}\int_0^R S_{13} r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^R a_4 C_{1323} x_1 - a_4 C_{1313} x_2 r \,\mathrm{d}r\,\mathrm{d}\theta \\ &= a_4\left[\underline{K_{2323}} - \underline{C_{1313}}\right]\int_0^{2\pi}\int_0^R r^3 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = 0 \end{aligned} \tag{3.23}$$

where $R$ is the radius of $\Gamma$.

Substitute (3.22) into the third equation of (2.6), then

$$\int_0^{2\pi}\int_0^R S_{33} r \,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^R [a_\rho x_\rho + a_3]\underline{S_{3333}}^{-1} r \,\mathrm{d}r\,\mathrm{d}\theta = -F_3$$

$$a_3 = \frac{-F_3 \underline{S_{3333}}}{\pi R^2} \tag{3.24}$$

Substitute (3.22) into the fourth equation of (2.6), then

$$\begin{aligned} &\int_0^{2\pi}\int_0^R [x_1 S_{23} - x_2 S_{13}] r \,\mathrm{d}r\,\mathrm{d}\theta = -M_3 \\ &a_4 \int_0^{2\pi}\int_0^R [x_1(x_1 C_{2323} - x_2 C_{1323}) - x_2(-x_2 C_{1313} + x_1 C_{1323})] r \,\mathrm{d}r\,\mathrm{d}\theta = -M_3 \end{aligned} \tag{3.25}$$

Substitute (B.1) into (3.25) and integrate, then

$$a_4 = \frac{-10 M_3}{\pi R^4\left(2R\underline{K_{1313}} + 2R\underline{K_{2323}} + 5\underline{C_{1313}}\right)} \tag{3.26}$$

Substitute (3.22) into the fifth and sixth equations of (2.6), then

$$\begin{aligned} \int_0^{2\pi}\int_0^R x_1 S_{33} r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^R x_1[a_\rho x_\rho + a_3]\underline{S_{3333}}^{-1} r \,\mathrm{d}r\,\mathrm{d}\theta = M_2 \\ \int_0^{2\pi}\int_0^R x_2 S_{33} r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^R x_2[a_\rho x_\rho + a_3]\underline{S_{3333}}^{-1} r \,\mathrm{d}r\,\mathrm{d}\theta = -M_1 \end{aligned} \tag{3.27}$$

Substitute (B.1) into (3.27) and integrate, then

$$a_1 = \frac{4}{\pi R^4} M_2 \underline{S_{3333}}, \quad a_2 = \frac{-4}{\pi R^4} M_1 \underline{S_{3333}} \tag{3.28}$$

Substitute (3.24), (3.26) and (3.28) into (3.22), then the stress functions become

$$\begin{aligned}
S_{11} &= S_{22} = S_{12} = 0 \\
S_{33} &= \frac{4x_1M_2}{\pi R^4} - \frac{4x_2M_1}{\pi R^4} - \frac{F_3}{\pi R^2} \\
S_{23} &= \frac{-10M_3}{\pi R^4\left(2R\underline{K_{1313}} + 2R\underline{K_{2323}} + 5\underline{C_{1313}}\right)}(C_{2323}x_1 - C_{1323}x_2) \\
S_{13} &= \frac{-10M_3}{\pi R^4\left(2R\underline{K_{1313}} + 2R\underline{K_{2323}} + 5\underline{C_{1313}}\right)}(C_{1323}x_1 - C_{1313}x_2)
\end{aligned} \tag{3.29}$$

## 4. Stress functions when considering $S_{3333}$ a function of $r$

There is a fundamental difference between the functions representing the shear stresses $(S_{23}, S_{13})$ and the normal stress $(S_{33})$ in Eq. (3.29). In Eq. (3.29) $S_{33}$ is independent of the material coefficients, while $S_{23}$ and $S_{13}$ depend on the material coefficients. Therefore, provided the constitutive equation is of the form (B.1), $S_{33}$ will only depend on the initial coordinates of a point and the applied loads, and not on the magnitudes of the material coefficients. However, $S_{23}$ and $S_{13}$ depend on the initial coordinates, the applied loads, and the magnitudes of the material coefficients, and so will vary for different materials even if the constitutive equation is similar to (B.1). The dependence of $S_{23}$ and $S_{13}$ on the material coefficients is a result of dependence of $S_{1313}$ and $S_{2323}$ (or $C_{1313}$ and $C_{2323}$) on the radial coordinate $r$. Thus, Section 4 will consider the constitutive equation where $S'_{3333}$ and $C'_{3333}$ also depend on $r$.

Let

$$\begin{aligned}
C'_{3333} &= \underline{C_{3333}} + r^*\underline{K_{3333}} \\
S'_{3333} &= \underline{S_{3333}} + r^*\underline{K_{3333}}
\end{aligned} \tag{4.1}$$

Recall that the transformation (2.2) is a rotation about the $x_3$ axis, therefore,

$$\begin{aligned}
C_{3333} &= \underline{C_{3333}} + r^*\underline{K_{3333}} \\
S_{3333} &= \underline{S_{3333}} + r^*\underline{K_{3333}}
\end{aligned} \tag{4.2}$$

Two important criteria must be satisfied in order to use Eq. (4.2) in the formulation developed in Section 3. First, the off diagonal terms in (2.11) were shown to equal zero when considering $C_{3333}$ a constant. Substituting (4.2) into (2.11) it can be seen that the off diagonal terms remain equal to zero, and so Eq. (2.12) is still valid. Second, in order to prove that $S_{\alpha\beta} = 0$ in (3.22) it was necessary for $C_{3333}(a_\rho x_\rho + a_3)$ to be analytic. By Eq. (4.2) $C_{3333}$ is only a function of $r$. Therefore, $C_{3333}(a_\rho x_\rho + a_3)$ is composed of analytic functions, coordinates, and constants and so must be analytic as well. Thus, the stress functions may be found using the formulation in Section 3, when substituting (4.2) into (B.1).

Recall the $S_{ij}$ stress functions

$$\begin{aligned}
S_{11} &= S_{22} = S_{12} = 0 \\
S_{33} &= S_{3333}^{-1}[a_\rho x_\rho + a_3] \\
S_{23} &= a_4C_{2323}x_1 - a_4C_{1323}x_2 \\
S_{13} &= a_4C_{1323}x_1 - a_4C_{1313}x_2
\end{aligned} \tag{3.22}$$

As in Section 3, substitution of (3.22) into the necessary conditions for a solution results in the first two equations of (2.6) being equal to zero for all $a_4$. For example

$$\int_0^{2\pi}\int_0^R S_{13} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^R a_4 C_{1323} x_1 - a_4 C_{1313} x_2 r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= a_4\left[\underline{K_{2323}} - \underline{C_{1313}}\right]\int_0^{2\pi}\int_0^R r^3\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{4.3}$$

Substitute (4.2) into (3.22) and this into the third equation of (2.6), then

$$\int_0^{2\pi}\int_0^R S_{33} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^R \left[a_\rho x_\rho + a_3\right]\left[\underline{S_{3333}} + r\underline{M_{3333}}\right]^{-1} r\,\mathrm{d}r\,\mathrm{d}\theta = -F_3$$

$$a_3 = \frac{-F_3\underline{M_{3333}^2}}{2\pi\left[R\underline{M_{3333}} - \underline{S_{3333}}\ln\left(\underline{S_{3333}} + R\underline{M_{3333}}\right) + \underline{S_{3333}}\ln\left(\underline{S_{3333}}\right)\right]} \tag{4.4}$$

Substitute (3.22) into the fourth equation of (2.6), then

$$\begin{aligned}&\int_0^{2\pi}\int_0^R [x_1 S_{23} - x_2 S_{13}] r\,\mathrm{d}r\,\mathrm{d}\theta = -M_3\\ &a_4\int_0^{2\pi}\int_0^R [x_1(x_1 C_{2323} - x_2 C_{1323}) - x_2(-x_2 C_{1313} + x_1 C_{1323})] r\,\mathrm{d}r\,\mathrm{d}\theta = -M_3\end{aligned} \tag{4.5}$$

Substitute (B.1) into (4.5) and integrate, then

$$a_4 = \frac{-10M_3}{\pi R^4\left(2R\underline{K_{1313}} + 2R\underline{K_{2323}} + 5\underline{C_{1313}}\right)} \tag{4.6}$$

Substitute (3.22) into the fifth and sixth equations of (2.6), then

$$\begin{aligned}&\int_0^{2\pi}\int_0^R x_1 S_{33} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^R x_1[a_\rho x_\rho + a_3]\left[\underline{S_{3333}} + r\underline{M_{3333}}\right]^{-1} r\,\mathrm{d}r\,\mathrm{d}\theta = M_2\\ &\int_0^{2\pi}\int_0^R x_2 S_{33} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^R x_2[a_\rho x_\rho + a_3]\left[\underline{S_{3333}} + r\underline{M_{3333}}\right]^{-1} r\,\mathrm{d}r\,\mathrm{d}\theta = -M_1\end{aligned} \tag{4.7}$$

Substitute (4.2) into (4.7) and integrate, then

$$\begin{aligned}a_1 &= \frac{6M_2\underline{M_{3333}^4}}{\pi\left[2R^3\underline{M_{3333}^3} - 3\underline{S_{3333}}R^2\underline{M_{3333}^2} + 6R\underline{S_{3333}^2}\underline{M_{3333}} + 6\underline{S_{3333}^3}\ln\left(\frac{\underline{S_{3333}}}{\underline{S_{3333}} + R\underline{M_{3333}}}\right)\right]}\\ a_2 &= \frac{-6M_1\underline{M_{3333}^4}}{\pi\left[2R^3\underline{M_{3333}^3} - 3\underline{S_{3333}}R^2\underline{M_{3333}^2} + 6R\underline{S_{3333}^2}\underline{M_{3333}} + 6\underline{S_{3333}^3}\ln\left(\frac{\underline{S_{3333}}}{\underline{S_{3333}} + R\underline{M_{3333}}}\right)\right]}\end{aligned} \tag{4.8}$$

The stress functions can be found by substituting (4.4), (4.6) and (4.8) into (3.22). All the nonzero stress functions in (3.22) are now dependent on the magnitudes of the material coefficients.

## 5. Summary

Consider the following four forms of the compliance tensor: (1) isotropic homogeneous, (2) orthotropic with respect to the cylindrical coordinates and homogeneous, (3) orthotropic with respect to the cylindrical coordinates and heterogeneous except for $S'_{3333}$, $S'_{1212}$ and $S'_{1122}$ being constant, and (4) orthotropic with respect to the cylindrical coordinates and heterogeneous except for $S'_{1212}$ and $S'_{1122}$ being constant.

The stress functions for a cylindrical section of a tree subject to loads independent of $x_3$ can be classified by the assumptions implicit in the compliance tensor (Table 1). When the compliance coefficients are

Table 1
The dependence of the stress functions on the compliance coefficients

| Constitutive equation | Magnitude of the in plane stresses $S_{11}$, $S_{22}$, and $S_{12}$ | Dependence of the nonzero stresses on the material coefficients |
|---|---|---|
| Isotropic homogeneous (Muskhelishvili, 1963) | Zero | All the stress functions are independent of the compliance coefficients |
| Orthotropic in cylindrical coordinates, homogeneous (Lyons et al., 2002a) | Zero | All the stress functions are independent of the compliance coefficients |
| Orthotropic in cylindrical coordinates, heterogeneous with $S'_{3333}$, $S'_{1212}$ and $S'_{1122}$ constant | Zero | All the stress functions, except for $S_{23}$ and $S_{13}$ are independent of the compliance coefficients |
| Orthotropic in cylindrical coordinates, heterogeneous with $S'_{1212}$ and $S'_{1122}$ constant | Zero | All the nonzero stress functions are dependent on the compliance coefficients |

allowed to depend on the cylindrical coordinate $r$ the stress functions become dependent on the compliance coefficients. Consider two beams with similar shape and size, and subject to similar loads. If the constitutive equations for both beams are dependent on $r$, but with dissimilar compliance coefficients, then the magnitudes of the stresses will be dissimilar. This is a dramatic change from the case where the compliance coefficients are homogeneous. If the compliance coefficients are homogeneous then the stresses will have similar magnitudes in the two beams.

## Appendix A. Transformation equations

The transformation equations taking the elasticity coefficients in cylindrical coordinates ($C'_{ijkl}$) to Cartesian coordinates ($C_{ijkl}$) (Lyons et al., 2002a).

$$C_{1111} = C_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2222}$$
$$C_{2222} = S_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + C_\theta^4 C'_{2222}$$
$$C_{3333} = C'_{3333}$$
$$C_{2323} = S_\theta^2 C'_{1313} + C_\theta^2 C'_{2323}$$
$$C_{1313} = C_\theta^2 C'_{1313} + S_\theta^2 C'_{2323}$$
$$C_{1212} = C_\theta^2 S_\theta^2 \left[C'_{1111} - 2C'_{1122} + C'_{2222} - 2C'_{1212}\right] + \left[C_\theta^4 + S_\theta^4\right] C'_{1212}$$
$$C_{1122} = C_\theta^2 S_\theta^2 C'_{1111} + C_\theta^4 C'_{1122} - 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2211} + C_\theta^2 S_\theta^2 C'_{2222}$$
$$C_{1133} = C_\theta^2 C'_{1133} + S_\theta^2 C'_{2233}$$
$$C_{1123} = 0$$
$$C_{1113} = 0$$
$$C_{1112} = -C_\theta S_\theta \left[C_\theta^2 C'_{1111} - C_\theta^2 C'_{1122} - 2C_\theta^2 C'_{1212} + 2S_\theta^2 C'_{1212} + S_\theta^2 C'_{1122} - S_\theta^2 C'_{2222}\right]$$
$$C_{2233} = S_\theta^2 C'_{1133} + C_\theta^2 C'_{2233}$$
$$C_{2223} = 0$$
$$C_{2213} = 0$$
$$C_{2212} = -C_\theta S_\theta \left[S_\theta^2 C'_{1111} - S_\theta^2 C'_{1122} - 2S_\theta^2 C'_{1212} + 2C_\theta^2 C'_{1212} + C_\theta^2 C'_{1122} - C_\theta^2 C'_{2222}\right]$$
$$C_{3323} = 0$$
$$C_{3313} = 0$$

$$
\begin{aligned}
&C_{3312} = -C_\theta S_\theta \left[C'_{3311} - C'_{3322}\right] \\
&C_{2313} = -C_\theta S_\theta \left[C'_{1313} - C'_{2323}\right] \\
&C_{2312} = 0 \\
&C_{1312} = 0
\end{aligned} \tag{A.1}
$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

For the transformation equations taking the compliance coefficients in cylindrical coordinates ($S'_{ijkl}$) to Cartesian coordinates ($S_{ijkl}$), replace $C'_{ijkl}$ with $S'_{ijkl}$ and $C_{ijkl}$ with $S_{ijkl}$ in Eq. (A.1).

## Appendix B. Transformed compliance coefficients

The transformation equations that transform the nonzero compliance coefficients in Eq. (2.1) ($S'_{ijkl}$) to Cartesian coordinates ($S_{ijkl}$) (Lyons et al., 2002b).

$$
\begin{aligned}
&S_{1111} = \lfloor C_\theta^4 + S_\theta^4 \rfloor \underline{S_{1111}} + r\lfloor C_\theta^4 \underline{M_{1111}} + S_\theta^4 \underline{M_{2222}} \rfloor + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
&S_{1122} = \underline{S_{1122}} \\
&S_{1133} = \underline{S_{1133}} + r\left[C_\theta^2 \underline{M_{1133}} + S_\theta^2 \underline{M_{2233}}\right] \\
&S_{1112} = -S_\theta C_\theta \left[C_\theta^2 \left[\underline{S_{1111}} + r\underline{M_{1111}}\right] - C_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + S_\theta^2 \underline{S_{1122}} - S_\theta^2 \left[\underline{S_{1111}} + r\underline{M_{2222}}\right]\right] \\
&S_{2222} = \left[C_\theta^4 + S_\theta^4\right] \underline{S_{1111}} + r\left[S_\theta^4 \underline{M_{1111}} + C_\theta^4 \underline{M_{2222}}\right] + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
&S_{2233} = \underline{S_{1133}} + r\left[S_\theta^2 \underline{M_{1133}} + C_\theta^2 \underline{M_{2233}}\right] \\
&S_{2212} = -S_\theta C_\theta \left[S_\theta^2 \left[\underline{S_{1111}} + r\underline{M_{1111}}\right] - S_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + C_\theta^2 \underline{S_{1122}} - C_\theta^2 \left[\underline{S_{1111}} + r\underline{M_{2222}}\right]\right] \\
&S_{3333} = \underline{S_{3333}} \\
&S_{3312} = -S_\theta C_\theta r\left[\underline{M_{1133}} - \underline{M_{2233}}\right] \\
&S_{2323} = \underline{S_{1313}} + r\left[S_\theta^2 \underline{M_{1313}} + C_\theta^2 \underline{M_{2323}}\right] \\
&S_{2313} = -S_\theta C_\theta r\left[\underline{M_{1313}} - \underline{M_{2323}}\right] \\
&S_{1313} = \underline{S_{1313}} + r\left[C_\theta^2 \underline{M_{1313}} + S_\theta^2 \underline{M_{2323}}\right] \\
&S_{1212} = C_\theta^2 S_\theta^2 \left[2\underline{S_{1111}} + r\left[\underline{M_{1111}} + \underline{M_{2222}}\right] - 2\left[\underline{S_{1122}} + \underline{S_{1212}}\right]\right] + \left[C_\theta^4 + S_\theta^4\right] \underline{S_{1212}}
\end{aligned} \tag{B.1}
$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

Similar equations can be formed for the elastic coefficients by replacing the $S_{ijkl}$ by $C_{ijkl}$, $\underline{S_{ijkl}}$ by $\underline{C_{ijkl}}$, and $\underline{M_{ijkl}}$ by $\underline{K_{ijkl}}$, in Eq. (B.1).

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